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Can one measure Hannay angles?

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Abstract. It has been overlooked that—with the exception of especially favourable circumstances—the answer seems to be negative. For single-frequency systems, however, an averaging procedure is shown to allow for an experimental investigation of the Hannay angles.

1. Introduction

The Hannay angles were introduced by Hannay [1] in order to draw attention to an anholonomy effect in classical mechanics closely corresponding to the Berry phase [2, 3] in quantum mechanics. Since the Hannay angles may be given a geometrical interpretation [4, 5] they are called *geometrical angles*.

In [4] we argued that the experimental verification of the Hannay angles is cumbersome, because it involves, in general, high-precision measurements, as these angles are typically small as compared to the *dynamical angles*. Our main point in this paper is that from the point of view of experimental investigations the dynamical angle shifts have been unduely neglected. As will be explained, in the case of a general system of several degrees of freedom, the Hannay angles seem not to be amenable to direct experimental measurement because one has insufficient control over the dynamical angle shifts. Nevertheless, even in cases where a direct measurement may not be possible, the Hannay angles remain an important theoretical concept.

The difficulty of measuring the Hannay angles also appears for single-frequency systems, but here we shall be able to settle the problem under certain circumstances by means of an averaging technique introduced in [6].

To be more explicit, we consider a classical Hamiltonian system depending on external parameters. Let this system, by fiat, be integrable for each fixed value of these parameters. In terms of action-angle variables the Hamiltonian can consequently be written as a function of the actions alone. If, however, we allow for a slow and periodic time dependence on the parameters, the dynamics in terms of action-angle variables (J, φ) will be determined by the Hamiltonian

$$h = h_0(J, \tau) + \varepsilon h_1(J, \varphi, \tau)$$

where εh_1 is just the time derivative of the generating function which governs the transformation to (J, φ). $\varepsilon > 0$ is a small quantity and $\tau := \varepsilon t$ is slow time. Because of the periodic time dependence of the parameters, h_0 and h_1 are T-periodic in τ

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for some positive T. Set $f(J,\varphi,\tau) := -\nabla_\varphi h_1(J,\varphi,\tau)$, $g(J,\varphi,\tau) := \nabla_J h_1(J,\varphi,\tau)$, and $\omega(J,\tau) := \nabla_J h_0(J,\tau)$; the Hamiltonian equations of motion are

$$\dot{J} = \varepsilon f(J,\varphi,\tau) \qquad \dot{\varphi} = \omega(J,\tau) + \varepsilon g(J,\varphi,\tau). \tag{1}$$

The *adiabatic Hannay angles* are defined for small $\varepsilon > 0$ by

$$\begin{aligned}\triangle\varphi &:= \varphi\left(\frac{T}{\varepsilon}\right) - \varphi^0 - \int_0^{T/\varepsilon} \omega(J(t),\varepsilon t)\,\mathrm{d}t \\ &= \varepsilon \int_0^{T/\varepsilon} g(J(t),\varphi(t),\varepsilon t)\,\mathrm{d}t\end{aligned} \tag{2}$$

where $\varphi^0 := \varphi(0)$.

In fact, we invoked averaging theory in [4, 6] to explain under what mathematically precisely defined circumstances the adiabatic Hannay angles may be replaced by the *geometrical Hannay angles*

$$\int_0^T g^{\mathrm{av}}(J^0,\tau)\,\mathrm{d}\tau \tag{3}$$

where $J^0 := J(0)$ and $g^{av}(J,\tau) := \int_0^{2\pi} \ldots \int_0^{2\pi} g(J,\varphi,\tau)\,d^n\varphi/(2\pi)^n$ and n is the number of degrees of freedom. It should be noted that in order to determine the geometrical Hannay angles one need not solve the Hamiltonian equations (1). It is precisely this fact that makes the averaged quantity (3) so useful, for the solution of the non-autonomous system (1) cannot, in general, be obtained analytically.

But how does one determine the Hannay angles experimentally? From (2) one reads off that a measurement of the Hannay angles involves an experimental determination of the angle $\varphi(T/\varepsilon)$. But this does not yet suffice, and in order to find the Hannay angles and to compare theory and experiment, the measurement has to be complemented by an *a priori* determination of the *dynamical angles* $\int_0^{T/\varepsilon} \omega(J(t),\varepsilon t)\,\mathrm{d}t$.

However, in general the dynamical angles cannot be easily estimated: neither can one solve (1) and determine $J(t)$ explicitly nor can one directly invoke averaging theory. In essence, averaging theory [7, 8] can be summarised by saying that (under suitable conditions such as, for instance, non-degeneracy) $\max_{t\in[0,T/\varepsilon]} |J(t) - J^0|$ is of the order $\mathcal{O}(\varepsilon)$ (as $\varepsilon \to 0$) in the case of a single degree of freedom, and of order $\mathcal{O}(\varepsilon^b)$, for any $b \in [0, \frac{1}{2})$, in the case of multi-frequency systems. Therefore the naive approximation $\int_0^{T/\varepsilon} \omega(J^0,\varepsilon t)\,\mathrm{d}t$ to the dynamical angles is too crude (as the the domain of integration has length T/ε). In fact, by means of first-order pertubation theory one can show for the simple example $h = J^2/2 + \varepsilon \cos\varphi$ that

$$\int_0^{T/\varepsilon} \left[\omega(J(t)) - \omega(J^0)\right]\,\mathrm{d}t = T\frac{\cos\varphi^0}{J^0} + \mathcal{O}(\varepsilon) \tag{4}$$

and the secular term prevents the right-hand side from vanishing as $\varepsilon \to 0$.

So, for this example, the error one obtains when replacing the time-evolved action by its initial value inside the dynamical angle is of the same order as the geometrical angle $\triangle\varphi$, i.e. $\mathcal{O}(1)$ (as $\varepsilon \to 0$). However, this error is of a different nature in so far as it is *not* geometrical.

On the other hand, (4) also suggests a remedy to the problem, namely averaging over the initial angles φ^0. In other words, upon averaging on the initial torus J^0 with the Liouville measure one may expect $\int_0^{T/\varepsilon} \omega(J^0, \varepsilon t)\, dt$ to be a good approximation to the dynamical angles. We will prove this expectation to be true for single-frequency systems.

Such an averaging procedure seems also to be acceptable from the experimental point of view provided the experimentalist has at his (or her) disposal the possibility of preparing the system again and again with one and the same J^0 and different φ^0. This premise, however, restricts the scope of experimental applicability of the proposed averaging over the initial torus and limits the extent to which our result has an impact on experimental measurements of Hannay angles. One such situation where the idea of averaging over initial angles seems not to be applicable is given, for instance, by the motion of celestial bodies, since in celestial mechanics one cannot choose initial conditions freely.

For general systems with several degrees of freedom the dilemma is even more severe, because our result on averaging over the initial torus will not carry over to them. This can be seen by means of a Taylor expansion of $\omega(J, \tau)$ up to second order with respect to J. Then the second-order term involves the square of the deviation of the action from the initial action, i.e. is of the order $\mathcal{O}(\varepsilon^{2b})$, for all $b \in [0, \frac{1}{2})$, and averaging over φ^0 will not improve the rate of convergence to zero, due to positivity of the square. The oscillations in φ^0 are of no help here: one can only obtain an approximation to the dynamical angles with an error of order $\mathcal{O}(\varepsilon^{2b-1})$ whilst the Hannay angles are an $\mathcal{O}(1)$ effect. Since $b < \frac{1}{2}$, in the adiabatic limit, the error term will exceed the Hannay angles. For such general systems it seems rather questionable whether there is an experimental method for determining the anholonomy effect discovered by Hannay. In §3 we will remark, however, on a special class of systems with several degrees of freedom where the results of this paper hold true [9].

One might also wonder whether it is possible to invoke Lenard's [10] and Neishtadt's [11] results on the accuracy of the conservation of the action in the single-frequency case. They proved that if a Hamiltonian of a system with one degree of freedom has a smooth time dependence that vanishes outside a finite time interval (or at infinity), then the overall variation of the action is small to any order in ε (or even exponentially). For finite times the action may undergo fluctuations of the order of ε, but this term is easily seen to be oscillating in time with zero mean. Therefore Lenard's and Neishtadt's result can be used to avoid averaging over initial angles. Their assumption that there be no time dependence of the Hamiltonian outside some time interval is, however, rather restrictive. In many situations one does not have at one's disposal an experimental change of the form of the time dependence.

As far as the Berry phase is concerned, the averaging procedure described in this paper is not necessary. Physically speaking, quantum mechanics already involves such an averaging because the phase space functions concentrate on the classically invariant phase space manifolds in the classical limit. In particular, for the ergodic case convergence to the Liouville measure was proven in [12]. Therefore, in quantum mechanics the dynamical phase shifts directly involve the energies $E_n(\tau)$ which arise from the stationary Schrödinger equation for *frozen* parameters, i.e. for fixed τ. However, solving this autonomous equation may, of course, present another difficulty.

Another interesting point concerning the measurement of angles was made by Vinti [13] in the case of integrable systems without additional time dependence. He points out that the angles φ are abstract angles and their frequencies are not directly observable in any other configuration space. However, for *conditionally periodic Stäckel systems* he shows that the time average of the frequencies in the configuration space chosen equals the frequencies of φ. Although his proof extends to the time-dependent situation considered in this paper, our averaging procedure must be carried out in the abstract angles φ^0.

We now summarise the paper. The results dealing with averaging over the initial torus in the single-frequency case (theorem 3 and corollary 4) will be stated and proven in §2. In §3 we give a few conclusions and remark on some special systems with several degrees of freedom as well as on future work [9].

2. Averaging over initial angles

Let us first fix the assumptions under which we will state our theorems. These conditions are essentially the same as in [6] except that we require the system to be Hamiltonian. This restriction is essential for theorem 3. We work on the phase space $M := D \times \mathbb{T}$, where D is a bounded open subset of $\mathbb{R}$ and $\mathbb{T}$ is the one-dimensional torus (of length 2π). The assumptions on the Hamiltonian $h = h_0(J,\tau) + \varepsilon h_1(J,\varphi,\tau)$ are

(i) h_0 and h_1 are in $C^2(M \times [0,T])$,

(ii) $\omega \neq 0$ on $M \times [0,T]$.

The second assumption excludes resonances.

First we will formulate the adiabatic theorem of classical (Hamiltonian) mechanics [7]. Let $B \subset D$ be an open subset of initial values of J^0. We assume that B has a strictly positive distance from the boundary ∂D of D.

Theorem 1. Under the above assumptions there exist $c_0 > 0$, $\varepsilon_0 > 0$ such that for any initial condition $(J^0, \varphi^0) \in B \times \mathbb{T}$, system (1) has a unique solution satisfying

$$\max_{0\leq t\leq T/\varepsilon} |J(t) - J^0| \leq c_0\varepsilon \qquad \forall\varepsilon \leq \varepsilon_0.$$

In [6] we proved an averaging theorem for phase space functions $a(J,\varphi,\tau)$. In the present (Hamiltonian) context this theorem holds when assuming a to be continuous and $a(\cdot,\cdot,\tau)$ to be $C^1(M)$ for all $\tau \in [0,T]$. Set

$$A(t) := \frac{\varepsilon}{T}\int_0^t a(J(u),\varphi(u),\varepsilon u)\,\mathrm{d}u \qquad \bar{A}(t) := \frac{\varepsilon}{T}\int_0^t a^{\mathrm{av}}(J^0,\varepsilon u)\,\mathrm{d}u$$

where $a^{\mathrm{av}}(J,\tau) := \int_0^{2\pi} a(J,\varphi,\tau)\,\mathrm{d}\varphi/2\pi$. $A(T/\varepsilon)$ and $\bar{A}(T/\varepsilon)$ denote, respectively, the time average of the phase space function $a(J,\varphi,\tau)$ and of its average over the (fast) angular variables.

Theorem 2. Under the above assumptions there exists $c_1 > 0$ such that for any initial condition $(J^0,\varphi^0) \in B \times \mathbb{T}$ one has

$$\max_{0\leq t\leq T/\varepsilon} \left|A(t) - \bar{A}(t)\right| \leq c_1\varepsilon \qquad \forall\varepsilon \leq \varepsilon_0.$$

Here ε_0 is taken from theorem 1.

Now we may state and prove a theorem crucial for the analysis of the averaged behaviour of the dynamical angles, where averaging refers to the initial angles φ^0. Such an average will be denoted by $\langle \cdot \rangle$.

Theorem 3. Let the above assumptions hold and asssume, in addition, that $h_1(\cdot,\cdot,\tau)$ is C^3 for all $\tau \in [0,T]$. Let $\Omega(J,\tau)$ be C^1 on $M \times [0,T]$. Then there exists $c_2 > 0$ such that for any initial action $J^0 \in B$,

$$\max_{0\le t\le T/\varepsilon} \left| \left\langle \int_0^t \Omega(J(u),\varepsilon u)\,\mathrm{d}u \right\rangle - \int_0^t \Omega(J^0,\varepsilon u)\,\mathrm{d}u \right| \le c_2\varepsilon \qquad \forall \varepsilon \le \varepsilon_0.$$

Again, ε_0 is taken from theorem 1.

Proof. Set

$$\triangle := \int_0^t \left[\Omega(J(u),\varepsilon u) - \Omega(J^0,\varepsilon u)\right]\,\mathrm{d}u$$

and let $\partial_J := \partial/\partial J$. Then the adiabatic theorem (theorem 1) gives, always assuming that $0 \le t \le T/\varepsilon$,

$$\begin{aligned}\triangle &= \int_0^t (J(u)-J^0)\partial_J\Omega(J^0,\varepsilon u) + \mathcal{O}(\varepsilon)\\ &= \varepsilon \int_0^t \left[\int_0^u f(J(v),\varphi(v),\varepsilon v)\,\mathrm{d}v\right] \partial_J\Omega(J^0,\varepsilon u)\,\mathrm{d}u + \mathcal{O}(\varepsilon).\end{aligned}$$

Here and in the following $\mathcal{O}(\varepsilon)$, $\varepsilon \to 0$, is always meant uniformly in $t \in [0,T/\varepsilon]$ and $J^0 \in B$.

To analyse $\int_0^u f\,\mathrm{d}v$ we employ the partial integration method developed in [6]. To this end we write f in terms of its Fourier series

$$f(J,\varphi,\tau) = \sum_{\nu\ne 0} f_\nu(J,\tau)\mathrm{e}^{\mathrm{i}\nu\varphi}$$

where ν runs over the (non-zero) integers in $\mathbb{Z}$. The Fourier coefficients are given by $f_\nu := \int_0^{2\pi} f(J,\varphi,\tau)\mathrm{e}^{-\mathrm{i}\nu\varphi}\,\mathrm{d}\varphi/2\pi$. A non-oscillatory term does not appear in the Fourier series of f since this function is a φ derivative. Set $\partial_v := \partial/\partial v$ etc. For $\nu \ne 0$, we have

$$\begin{aligned}\int_0^u f_\nu \mathrm{e}^{\mathrm{i}\nu\varphi}\,\mathrm{d}v &= \frac{1}{\mathrm{i}\nu}\int_0^u \frac{f_\nu}{\omega}\mathrm{e}^{\mathrm{i}\nu\varphi^0}\partial_v\left[\exp\left(\mathrm{i}\nu\int_0^v \omega\,\mathrm{d}w\right)\right]\exp\left(\mathrm{i}\varepsilon\nu\int_0^v g\,\mathrm{d}w\right)\,\mathrm{d}v\\ &= \left[\frac{f_\nu}{\mathrm{i}\nu\omega}\mathrm{e}^{\mathrm{i}\nu\varphi}\right]_{v=0}^{v=u} - \varepsilon\int_0^u \frac{1}{\omega} f_\nu g\,\mathrm{e}^{\mathrm{i}\nu\varphi}\,\mathrm{d}v\\ &\quad - \frac{\varepsilon}{\mathrm{i}\nu}\int_0^u \frac{1}{\omega}\left(f\partial_J f_\nu + \partial_\tau f_\nu - f_\nu f\frac{\partial_J\omega}{\omega} - f_\nu\frac{\partial_\tau\omega}{\omega}\right)\mathrm{e}^{\mathrm{i}\nu\varphi}\,\mathrm{d}v.\end{aligned}$$

The absolute convergence of the Fourier series is uniform on $K \times [0,T]$, for any compact K with $B \subset K \subset D$. One can convince oneself of this fact by adapting the usual proof of absolute convergence of the Fourier series of a periodic C^1 function. Therefore the order of integration and summation may be exchanged and

$$\int_0^u f\,\mathrm{d}v = \sum_{\nu\neq 0} \frac{1}{\mathrm{i}\nu}\left[\frac{f_\nu}{\omega}e^{\mathrm{i}\nu\varphi}\right]_{v=0}^{v=u} - \varepsilon\int_0^u \frac{1}{\omega} f g\,dv$$
$$-\varepsilon\sum_{\nu\neq 0}\frac{1}{\mathrm{i}\nu}\int_0^u \frac{1}{\omega}\left(f\partial_J f_\nu + \partial_\tau f_\nu - f_\nu f\frac{\partial_J\omega}{\omega} - f_\nu\frac{\partial_\tau\omega}{\omega}\right)\mathrm{e}^{\mathrm{i}\nu\varphi}\,\mathrm{d}v. \tag{5}$$

Note that both $\partial_J f_\nu/\nu$ and $\partial_\tau f_\nu/\nu$ are Fourier coefficients of Fourier series whose absolute convergence is uniform in (J,τ). In fact, if $c(J,\varphi,\tau)$ is a continuous function, C^1 and 2π periodic in φ, and of zero mean, then $d(J,\varphi,\tau) := \int_0^\varphi c(J,\theta,\tau)\,\mathrm{d}\theta$ is also 2π periodic in φ and $d(J,\varphi,\tau) = d^{\mathrm{av}}(J,\tau) + \sum_{\nu\neq 0} c_\nu(J,\tau)\mathrm{e}^{\mathrm{i}\nu\varphi}/\mathrm{i}\nu$.

Now we want to make sure that the v integration in (5) extends only over oscillatory terms. Clearly, by the Hamiltonian character of (1), $\partial_J f_\nu = -\mathrm{i}\nu g_\nu$. By the remark of the last paragraph,

$$\sum_{\nu\neq 0}\frac{1}{\mathrm{i}\nu}\int_0^u \frac{1}{\omega} f\partial_J f_\nu \mathrm{e}^{\mathrm{i}\nu\varphi}\,\mathrm{d}v + \int_0^u \frac{1}{\omega} f g\,\mathrm{d}v = \int_0^u \frac{1}{\omega} f g^{\mathrm{av}}\,\mathrm{d}v.$$

Moreover,

$$\sum_{\nu\neq 0}\frac{1}{\nu} f_\nu f_{-\nu} = \sum_{\nu>0}\frac{1}{\nu}\left(f_\nu f_{-\nu} - f_{-\nu} f_\nu\right) = 0$$

and thus $f\sum_{\nu\neq 0} f_\nu \mathrm{e}^{\mathrm{i}\nu\varphi}/\nu$ does not contain any non-oscillatory terms.

All in all, we obtain

$$\int_0^u f\,dv = \sum_{\nu\neq 0}\frac{1}{\mathrm{i}\nu}\left[\frac{f_\nu}{\omega}\mathrm{e}^{\mathrm{i}\nu\varphi}\right]_{v=0}^{v=u} + \mathcal{O}(\varepsilon)$$

after having applied theorem 2 to mean zero phase space functions. It is here where we need $h_1(\cdot,\cdot,\tau)$ to be C^3. A second application of this theorem (now to the boundary term with $v = u$) yields

$$\triangle = \varepsilon\sum_{\nu\neq 0}\frac{\mathrm{i}f_\nu(J^0,0)}{\nu\omega(J^0,0)}\mathrm{e}^{\mathrm{i}\nu\varphi^0}\int_0^t \partial_J\Omega(J^0,\varepsilon u)\,\mathrm{d}u + \mathcal{O}(\varepsilon).$$

Therefore $\langle\triangle\rangle = \mathcal{O}(\varepsilon)$. □

In fact, by inspection of the proofs it becomes obvious that one can weaken the regularity assumptions in above theorems. For instance, one order less of differentiability with respect to φ will also suffice.

As far as the averaged time evolution of the dynamical angle is concerned, theorem 3 has the following consequence.

Corollary. Let $h = h_0 + \varepsilon h_1$ satisfy the assumptions of theorem 3. Then there exists $c_3 > 0$ such that for any initial action $J^0 \in B$,

$$\max_{0\le t\le T/\varepsilon}\left|\left\langle\int_0^t \omega(J(u),\varepsilon u)\,\mathrm{d}u\right\rangle - \int_0^t \omega(J^0,\varepsilon u)\,\mathrm{d}u\right| \le c_2\varepsilon \qquad \forall\varepsilon\le\varepsilon_0.$$

Here ε_0 is taken from theorem 1.

This corollary substantiates our claim that for single-frequency systems one may replace, upon averaging over the initial torus and up to a small error term, the full action inside the averaged dynamical angle by its initial value, thus yielding the possibility of an experimental determination of the Hannay angle for such systems (whenever the averaging is experimentally feasible).

3. Conclusion

As we have shown, in the adiabatic limit the dynamical angles of systems with a single degree of freedom are approximated—upon averaging over the initial torus J^0—by the simpler quantity $\int_0^{T/\varepsilon} \omega(J^0, \varepsilon t)\, \mathrm{d}t$. It was motivated in the introduction that such a statement will, in general, be false for systems with several degrees of freedom. This is essentially due to the occurence of resonances.

Let us mention two examples with several degrees of freedom where one would nevertheless expect the situation to be not as bad. The first is a satellite in the gravitational field of a slowly rotating oblate (or prolate) planet [4]. And, second, Kugler [14] investigated a vibrating string set at an angle with respect to the axis of a slowly rotating base (a Foucault pendulum in disguise, as he calls it); cf also [15].

Common to both examples is that in terms of coordinates with respect to co-rotating frames of reference there appears no explicit time dependence in the unperturbed Hamiltonian h_0, i.e. $h_0 = h_0(J)$. In subsequent work [9], it will be argued that if the unperturbed Hamiltonian h_0 is not explicitly time dependent then also for multi-frequency systems a result in the spirit of theorem 3 holds true for a large (in the sense of measure) set of initial values. Basically, such systems are not driven into resonances by an outside time dependence and, consequently, pertubation theory allows one to prove adiabatic invariance up to order $\mathcal{O}(\varepsilon^\beta)$, for all $\beta \in [0, 1)$.

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